New Two-Dimensional Quantum Models With Shape Invariance

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Introduction

We shall consider intertwining of two scalar Hamiltonians $H^{(1)}$, $H^{(2)}$ in the two-dimensional case by supercharges $Q^{\pm}$ of second order in derivatives

$$H^{(1)}(\vec{x})Q^{+}(\vec{x}) = Q^{+}(\vec{x})H^{(2)}(\vec{x}), \tag{1}$$

$$Q^{-}(\vec{x})H^{(1)}(\vec{x}) = H^{(2)}(\vec{x})Q^{-}(\vec{x}), \tag{2}$$

$$H^{(1),(2)} = -\partial_1^2 - \partial_2^2 + V^{(1),(2)}(\vec{x});\ \vec{x} = (x_1, x_2),$$

where supercharges are chosen with Lorentz metrics in second order derivatives:

$$Q^{+} = 4\partial_{+}\partial_{-} + 4C_{+}\partial_{-} + 4C_{-}\partial_{+} + B(\vec{x});\ Q^{-} = (Q^{+})^{\dagger};\ x_{\pm} = x_1 \pm x_2. \tag{3}$$

As was shown in A.Andrianov, M.Ioffe, D.N., 1995, 1996, the solution of intertwining relations (1), (2) leads to $C_\pm = C_\pm(x_\pm)$ and to the expressions for the potentials:

$$V^{(1,2)}(\vec{x}) = 2(C'_+ \pm C^2_+ + C'_- \pm C^2_-) + f_2(x_2) - f_1(x_1) \equiv$$
$$\equiv v_+^{(1,2)}(x_+) + v_-^{(1,2)}(x_-) + f_2(x_2) - f_1(x_1), \qquad (4)$$

where the functions $C_\pm(x_\pm)$ and $F(\vec{x}) \equiv f_1(x_1) + f_2(x_2)$ have to satisfy the following equation:

$$\partial_-(C_- F) = -\partial_+(C_+ F), \qquad (5)$$

This equation was solved by means of choosing some simplifying ansatzes. Some of these solutions obey two-dimensional shape invariance property.

We shall study the general question: is there any, wide enough, additional class of shape invariant systems? In one-dimensional Quantum Mechanics with first order differential operators $Q^{\pm}$, one refers to shape invariance Gendenshtein, 1983, when the partner Hamiltonians in (1), (2) both depend on some (multi) parameter a, and they have the similar shape, i.e. they satisfy:

$$H^{(2)}(x,a) = H^{(1)}(x,\tilde{a}) + \mathcal{R}(a), \tag{6}$$

where $\tilde{a} = \tilde{a}(a)$ is some new value of parameter, which depends on a, and $\mathcal{R}(a)$ is a (c-number) function of a.

Two-dimensional shape invariance

The problem of constructing of most general form of shape invariant potential seems to be a rather difficult task. Even in one-dimensional SUSY Quantum Mechanics it is not yet fully solved. Only a class of such potentials was built, and it coincides with variety of well known exactly solvable potentials. It was proven recently J.Bougie, A.Gangopadhyaya, J.V.Mallow, 2010 that no additional shape invariant potentials with so called additive shape invariance exist, outside this class.
It is clear that one has not too many chances to solve the analogous problem in two-dimensional situation. Therefore, we do not pretend to find all existing shape invariant two-dimensional potentials. At best, we can find a wide class of such new potentials.

It is necessary to remind that some shape invariant potentials were already obtained A.Andrianov,F.Cannata, M.Ioffe, D.N., 1995-2002

in the framework of polynomial SUSY QM. Among these potentials two-dimensional generalizations of Morse potential and Pöschl-Teller potential must be mentioned specially M.Ioffe, D.N., 2007, M.Ioffe, P.Valinevich, 2005.

It is convenient, without loss of generality, to choose parameters in such a way that the shape invariance condition (6) links Hamiltonians $H^{(1,2)}(\vec{x}; a)$ with difference (step) between parameters equal to unity, i.e.

$$H^{(1)}(a+1; \vec{x}) - H^{(2)}(a; \vec{x}) = const. \tag{7}$$

It is evident from the explicit expressions (4) for potentials, that the shape invariance (7) is equivalent to a pair of one-dimensional shape invariance with "superpotentials" $C_\pm(x_\pm)$:

$$v_\pm^{(1)}(a_\pm + 1; x_\pm) - v_\pm^{(2)}(a_\pm; x_\pm) = c_\pm, \qquad (8)$$

i.e. Hamiltonians $H^{(1,2)}$ obey actually two-parametric shape invariance with the two-component parameter $a \equiv (a_+,\, a_-)$. Besides $v_\pm^{(1,2)}$, the potentials $V^{(1,2)}(a; \vec{x})$ contain also the terms $f_2(x_2) - f_1(x_1)$, which prevent separation of variables.
The dependence of $H^{(1,2)}$ on a seems superficially to be a generic dependence on two independent variables $a_+,\, a_-$.
However, as will be clear from the following Eq.(12) there is a constraint.

It is well known A.Andrianov, M.Ioffe, D.N., 1995, 1996, that arbitrary Hamiltonians $H^{(1)}(a;\vec{x}), H^{(2)}(a;\vec{x})$, which participate in SUSY intertwining relation, are integrable, i.e. they obey the symmetry operators (here of fourth order in derivatives):

$$[R^{(2)}(a;\vec{x}), H^{(2)}(a;\vec{x})] = 0, \quad R^{(2)}(a;\vec{x}) = Q^{+}(a;\vec{x})Q^{-}(a;\vec{x})$$
$$[R^{(1)}(a;\vec{x}), H^{(1)}(a;\vec{x})] = 0; \quad R^{(1)}(a;\vec{x}) = Q^{-}(a;\vec{x})Q^{+}(a;\vec{x}).$$

The shape invariance relation (7) allows to conclude that:

$$[R^{(1)}(a+1;\vec{x}) - R^{(2)}(a;\vec{x}), H^{(2)}(a;\vec{x})] = 0. \tag{9}$$

Although the operator $R^{(1)}(a+1;\vec{x}) - R^{(2)}(a;\vec{x})$ seems to be a symmetry operator for $H^{(2)}(a;\vec{x})$, one can check that it is of second order in derivatives.

The symmetry operator of second order in momenta signals that the Hamiltonian is amenable to separation of variables W.Miller, Jr., 1977 Since from the very beginning, we do not consider systems with separation of variables, we shall concern ourselves only with the case when this operator is a function of $H^{(2)}(a;\vec{x})$, i.e.:

$$R^{(1)}(a+1;\vec{x}) - R^{(2)}(a;\vec{x}) = \mu H^{(2)}(a;\vec{x}) + \nu, \tag{10}$$

with μ, ν-constants.
Using expressions for $Q^{\pm}$, the l.h.s. of (10) can be calculated explicitly:

$$R^{(1)}(a+1) - R^{(2)}(a) = -8\left(v_+^{(1)}(a_+ + 1) - v_+^{(2)}(a_+)\right)\partial_-^2 - -8\left(v_-^{(1)}(a_- + 1) - v_-^{(2)}(a_-)\right)\partial_+^2 + R(a,\vec{x}), \tag{11}$$

where the $R(a,\vec{x})$ is function.

Comparing with (10), we obtain the necessary condition, which constants $c_\pm$ in the r.h.s of Eq.(8) have to satisfy. Namely they should coincide:

$$v_+^{(1)}(a_+ + 1) - v_+^{(2)}(a_+) = v_-^{(1)}(a_- + 1) - v_-^{(2)}(a_-) = c. \qquad (12)$$

It was shown in J.Bougie, A.Gangopadhyaya, J.V.Mallow, 2010, that it is sufficient to use the two variants of "superpotentials" $C_\pm$ on parameters $a_\pm$:

$$I) \qquad C_\pm(a_\pm) = a_\pm p_\pm + r_\pm; \qquad (13)$$

$$II) \qquad C_\pm(a_\pm) = a_\pm p_\pm + q_\pm(a_\pm). \qquad (14)$$

where $a_\pm$ are parameters of shape invariance with a unit step $\widetilde{a}_\pm = a_\pm + 1$, functions $p_\pm(x_\pm), r_\pm(x_\pm)$ do not depend on $a_\pm$, and $q_\pm(a_\pm)$ do not depend on $x_\pm$.

Construction of shape invariant potentials

The shape invariance of first kind can be considered, starting from Eq.(12), which for the choice (13) reads as:

$$(2a_{\pm}+1)(p_{\pm}^2+p_{\pm}')+2p_{\pm}r_{\pm}+2r_{\pm}'=c/2, \qquad (15)$$

where c is an arbitrary constant, the same for signs $\pm$ in the l.h.s. It follows that in this case, $p_{\pm}(x_{\pm})$, $r_{\pm}(x_{\pm})$ satisfy the system of differential equations:

$$p_{\pm}^2+p_{\pm}'=\lambda_{\pm}^2; \quad p_{\pm}r_{\pm}+r_{\pm}'=d_{\pm},$$

and its general solution is:

$$p_{\pm} = \frac{z'_{\pm}(x_{\pm})}{z_{\pm}(x_{\pm})};\ z_{\pm}(x_{\pm}) = \sigma_{\pm}\exp\left(\lambda_{\pm}x_{\pm}\right) + \delta_{\pm}\exp\left(-\lambda_{\pm}x_{\pm}\right); \tag{16}$$

$$r_{\pm} = \frac{1}{z_{\pm}(x_{\pm})}\left(\alpha_{\pm} + \frac{d_{\pm}}{\lambda_{\pm}^{2}} z'_{\pm}(x_{\pm})\right), \tag{17}$$

with integration constants $\alpha_{\pm}$. The form (13) of $C_{\pm}$ and expressions (16), (17) in terms of z, z' give an to take $d_{\pm} = 0$ by means of transformation of parameter $a_{\pm}$ to $(a_{\pm} + d_{\pm}/\lambda_{\pm}^{2})$. Thus, we shall continue with $d_{\pm} = 0$ below. Also, the direct calculations give that the constant c in the r.h.s. of Eq.(12) is: $c = 2(2a_{\pm} + 1)\lambda_{\pm}^{2}$, and therefore $a_{+} = a_{-} \equiv a$ and $\lambda_{+} = \lambda_{-} \equiv \lambda$. The case $\lambda_{+} = -\lambda_{-}$ is obtained by replacing $\sigma_{-} \leftrightarrow \delta_{-}$.

Now we can go to the system of equations (5). Since the function F does not depend on a, it follows from (5):

$$\partial_+(Fp_+) + \partial_-(Fp_-) = 0; \qquad (18)$$
$$\partial_+(Fr_+) + \partial_-(Fr_-) = 0. \qquad (19)$$

If one of the constants α_+ or α_- in (17) vanishes, it is clear from (19) that F is factorizable $F = Const/(f_+(x_+)f_-(x_-))$. This case was studied in A.Andrianov, M.Ioffe, D.Nishnianidze, 1995 in a general form. One of two solutions, which were found there, indeed obeys shape invariance, but it is amenable to standard separation of variables. By this reason, we shall not consider this case further we shall study two other choices of constants:

$$Ia) \quad \alpha_- = \alpha_+ = 0; \qquad Ib) \quad \alpha_-\alpha_+ \neq 0.$$

Ia). For this option, Eq.(19) is satisfied identically, but we have no direct way to solve Eq.(18) in a general form. We shall act in an indirect way. The function of $R(a, \vec{x})$ **in (11) is:**

$$R(a, \vec{x}) = 2(2a+1)\Big(4\lambda^2(f_1 - f_2) + (p_+ + p_-)f_1' + (p_- - p_+)f_2' + \\ +4p_+p_-(f_1 + f_2)\Big) + M(\vec{x}). \tag{20}$$

According to Eq.(10) and since the functions $f_{1,2}, p_\pm$ **do not depend on parameter** a, **each of last two terms in (20) must be constant. In particular,**

$$4\lambda^2(f_1 - f_2) + (p_+ + p_-)f_1' + (p_- - p_+)f_2' + 4p_+p_-(f_1 + f_2) \equiv 2\omega; \ \omega = const.$$

Together with (18), this equation gives:

$$f_2(x_2) = \frac{4\lambda^2 f_1(x_1) - \omega + (p_+ + p_-) f_1'(x_1)}{(p_+ - p_-)^2} - f_1(x_1), \qquad (21)$$

and after differentiation over x_1, **we obtain the equation for the function** $f_1(x_1)$:

$$f_1'' + \frac{3z_1'}{z_1} f_1' + 8\lambda^2 f_1 = 2\omega, \qquad (22)$$

$$z_1(x_1) \equiv \sigma_+ \sigma_- \exp(2\lambda x_1) - \delta_+ \delta_- \exp(-2\lambda x_1),$$

whose solution is:

$$f_1(x_1) = \frac{\omega}{4\lambda^2} + \frac{k_1 z_1' + k_2}{z_1^2}, \qquad (23)$$

with k_1, k_2- **constants. Substitution back into (21) gives:**

$$f_2(x_2) = -\frac{\omega}{4\lambda^2} + \frac{k_1 z_2' - k_2}{z_2^2}, \qquad (24)$$

with the definition of z_2 :

$$z_2(x_2) \equiv \sigma_+\delta_- \exp\left(2\lambda x_2\right) - \delta_+\sigma_- \exp\left(-2\lambda x_2\right). \qquad (25)$$

It follows from (23) and (24), that we may take the value $\omega = 0$, since f_1 and f_2 are defined up to an additive constant with opposite sign. Thus, the option Ia) gives the following shape invariant potentials:

$$V^{(1),(2)}(a;\vec{x}) = -8a(a\mp 1)\lambda^2\left(\frac{\sigma_+\delta_+}{(\sigma_+\exp(\lambda x_+) + \delta_+\exp(-\lambda x_+))^2} + \right.$$
$$\left. + \frac{\sigma_-\delta_-}{(\sigma_-\exp(\lambda x_-) + \delta_-\exp(-\lambda x_-))^2}\right) -$$
$$-\frac{k_1(\sigma_+\sigma_-\exp(2\lambda x_1) + \delta_+\delta_-\exp(-2\lambda x_1)) + k_2}{(\sigma_+\sigma_-\exp(2\lambda x_1) - \delta_+\delta_-\exp(-2\lambda x_1))^2} +$$
$$+\frac{k_1(\sigma_+\delta_-\exp(2\lambda x_2) + \delta_+\sigma_-\exp(-2\lambda x_2)) - k_2}{(\sigma_+\delta_-\exp(2\lambda x_2) - \delta_+\sigma_-\exp(-2\lambda x_2))^2}. \qquad (26)$$

Ib). In this case we shall solve the system of equations (18), (19) directly. This task is simplified by assuming $\alpha_+ = \alpha_- \equiv \alpha$ in (17) without loss of generality. After simple manipulations we obtain the system (18), (19) in the form:

$$\partial_1(\ln F(p_+ r_- - r_+ p_-)) = \frac{\lambda^2(r_+ - r_-)}{p_+ r_- - r_+ p_-},$$

$$\partial_2(\ln F(p_+ r_- - r_+ p_-)) = -\frac{\lambda^2(r_+ + r_-)}{p_+ r_- - r_+ p_-},$$

and using expressions (16), (17):

$$\partial_1(\ln F(p_+ r_- - r_+ p_-)) = -\lambda^2 \frac{z_+ - z_-}{z'_+ - z'_-} = -\partial_1 \ln(z'_+ - z'_-).$$

$$\partial_2(\ln F(p_+ r_- - r_+ p_-)) = -\lambda^2 \frac{z_+ + z_-}{z'_+ - z'_-} = -\partial_2 \ln(z'_+ - z'_-).$$

These equations can be integrated explicitly:

$$F = \frac{z_+ z_-}{(z'_+ - z'_-)^2}, \tag{27}$$

but it is necessary to take into account additionally that $F = f_1(x_1) + f_2(x_2)$. **This gives restriction onto parameters in (16):** $\sigma_+\delta_+ = \sigma_-\delta_-$. **Then, for** $\delta_+ \neq 0$, **we obtain from (27). that:**

$$f_1 = \frac{4k\sigma_-}{(\sigma_- \exp(\lambda x_1) + \delta_+ \exp(-\lambda x_1))^2},$$
$$f_2 = \frac{4k\delta_-}{(\delta_- \exp(\lambda x_2) - \delta_+ \exp(-\lambda x_2))^2},$$

and corresponding shape invariant potentials are:

$$V^{(1),(2)}(a;\vec{x}) = -4\left(2\lambda^2 a(a\mp 1)\sigma_-\delta_- - \alpha^2\right)\cdot$$

$$\cdot\left(\frac{\delta_+^2}{(\sigma_-\delta_-\exp(\lambda x_+) + \delta_+^2\exp(-\lambda x_+))^2} + \right.$$

$$\left.\frac{1}{(\sigma_-\exp(\lambda x_-) + \delta_-\exp(-\lambda x_-))^2}\right)$$

$$+4\alpha(2a\mp 1)\lambda\left(\frac{\delta_+((\sigma_-\delta_-\exp(\lambda x_+) - \delta_+^2\exp(-\lambda x_+))}{(\sigma_-\delta_-\exp(\lambda x_+) + \delta_+^2\exp(-\lambda x_+))^2} + \right.$$

$$\left.+\frac{\sigma_-\exp(\lambda x_-) - \delta_-\exp(-\lambda x_-)}{(\sigma_-\exp(\lambda x_-) + \delta_-\exp(-\lambda x_-))^2}\right) +$$

$$+4k\left(\frac{\delta_-}{(\delta_-\exp(\lambda x_2) - \delta_+\exp(-\lambda x_2))^2} - \right.$$

$$\left.-\frac{\sigma_-}{(\sigma_-\exp(\lambda x_1) + \delta_+\exp(-\lambda x_1))^2}\right). \tag{28}$$